Capstone Project Part 1: Option Pricing, FTC, and Taylor Approximation

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Project Parameters

$S_0 = 7414$	SPX index level (May 13, 2026)
$K = 7900$	strike (index points)
$T = 0.523$	years to expiry (Nov 20, 2026)
$r = 0.036$	13-week T-bill rate
$C_{\text{market}} = \$180.30$	bid-ask midpoint

Part 1(a) — Estimate σ from Historical Data

Choice. `lookback_days = 2018` ($\approx$ 8 calendar years of trading history).

Rationale. A 2018-day window covers the period from May 2018 through May 2026. This is long enough to deliver a statistically stable estimate ($n \approx 2017$ daily returns) and to span multiple market regimes — the calm of 2019, the COVID crash and recovery of 2020, the 2022 bear market, and the 2023–2026 rally. A shorter window (e.g. 90 or 180 days) is dominated by whatever regime is current and is noisy; a much longer window risks contaminating the estimate with structural conditions that no longer apply. Eight years is a common practitioner default and balances stability against recency.

Output from the notebook.

Calendar days requested:	2018
Daily returns computed:	2017
Daily std. dev. of log-returns:	0.012257
Annualized volatility σ:	**0.1946** (19.46%)

Annualization uses $\sigma = s_{\text{daily}} \cdot \sqrt{252}$ where 252 is the number of trading days per year and the $\sqrt{\cdot}$ comes from the additivity of variance for independent daily returns.

Part 1(b) — Compute $\Phi(d_1)$ and $\Phi(d_2)$ Using the Taylor Series

(i) Taylor series for e^x centered at $x = 0$

Because $\frac{d^n}{dx^n} e^x \big|_{x=0} = 1$ for every n,

$$e^x \;=\; 1 \;+\; x \;+\; \frac{x^2}{2!} \;+\; \frac{x^3}{3!} \;+\; \frac{x^4}{4!} \;+\; \cdots \;=\; \sum_{n=0}^{\infty} \frac{x^n}{n!}.$$

(ii) Substitute $x = -t^2/2$ to obtain the series for $e^{-t^2/2}$

Replacing x with $-t^2/2$:

$$e^{-t^2/2} \;=\; \sum_{n=0}^{\infty} \frac{(-t^2/2)^n}{n!} \;=\; \sum_{n=0}^{\infty} \frac{(-1)^n\, t^{2n}}{2^n\, n!}.$$

First five terms (each coefficient simplified):

$$e^{-t^2/2} \;=\; 1 \;-\; \frac{t^2}{2} \;+\; \frac{t^4}{8} \;-\; \frac{t^6}{48} \;+\; \frac{t^8}{384} \;-\; \cdots$$

because $2^0 \cdot 0! = 1$, $2^1 \cdot 1! = 2$, $2^2 \cdot 2! = 8$, $2^3 \cdot 3! = 48$, $2^4 \cdot 4! = 384$.

(iii) Integrate term by term to get $\Phi(z)$

By the Fundamental Theorem of Calculus, a power series may be integrated term by term inside its interval of convergence (here, all of $\mathbb{R}$). For $z \geq 0$ write

$$\Phi(z) \;=\; \frac{1}{2} \;+\; \int_0^z \varphi(t)\, dt \;=\; \frac{1}{2} \;+\; \frac{1}{\sqrt{2\pi}} \int_0^z e^{-t^2/2}\, dt,$$

using the fact that φ is even, so $\int_{-\infty}^0 \varphi(t)\, dt = \frac{1}{2}$. (The same formula holds for $z < 0$ since the antiderivative is odd.) Integrating each monomial via $\int_0^z t^{2n}\, dt = z^{2n+1}/(2n+1)$:

$$\boxed{\Phi(z) \;=\; \frac{1}{2} \;+\; \frac{1}{\sqrt{2\pi}} \sum_{n=0}^{\infty} \frac{(-1)^n\, z^{2n+1}}{2^n\, n!\,(2n+1)}.}$$

Writing out the first few terms,

$$\Phi(z) \;=\; \frac{1}{2} \;+\; \frac{1}{\sqrt{2\pi}} \left[z \;-\; \frac{z^3}{6} \;+\; \frac{z^5}{40} \;-\; \frac{z^7}{336} \;+\; \frac{z^9}{3456} \;-\; \cdots \right].$$

The constant $1/\sqrt{2\pi} \approx 0.3989423$ will be used in the numerical evaluation.

(iv) Compute d_1, d_2, then evaluate $\Phi(d_1)$ and $\Phi(d_2)$

Every intermediate calculation below is rounded to four decimal places before being used in the next step.

Step 1: Build the pieces of d_1

With $\sigma = 0.1946$:

$$\begin{aligned}
\sigma^2 &= (0.1946)^2 = 0.0379,\\
\sigma^2/2 &= 0.0379/2 = 0.0190,\\
r + \sigma^2/2 &= 0.036 + 0.0190 = 0.0550,\\
(r + \sigma^2/2)T &= 0.0550 \times 0.523 = 0.0288,\\
S_0/K &= 7414/7900 = 0.9385,\\
\ln(S_0/K) &= \ln(0.9385) = -0.0635,\\
\sqrt{T} &= \sqrt{0.523} = 0.7232,\\
\sigma\sqrt{T} &= 0.1946 \times 0.7232 = 0.1407.
\end{aligned}$$

Step 2: Assemble d_1 and d_2

$$\begin{aligned}
d_1 \;&=\; \frac{\ln(S_0/K) + (r + \sigma^2/2)\,T}{\sigma\sqrt{T}} \;=\; \frac{-0.0635 + 0.0288}{0.1407} \;=\; \frac{-0.0347}{0.1407} \;=\; -0.2466.\\
d_2 \;&=\; d_1 - \sigma\sqrt{T} \;=\; -0.2466 - 0.1407 \;=\; -0.3873.
\end{aligned}$$

Step 3: Evaluate $\Phi(d_1)$ for $d_1 = -0.2466$

Compute successive powers (each rounded to 4 dp) and the series terms (stop when the next term has absolute value < 0.0001):

$$\begin{aligned}
z &= -0.2466\\
z^2 &= 0.0608\\
z^3 &= -0.0150\\
z^5 &= -0.0009\\
z^7 &= -0.0001
\end{aligned}$$

$$\begin{aligned}
T_0 = z \quad &= -0.2466, \quad |T_0| = 0.2466\\
T_1 = -\,z^3/6 \quad &= +0.0025, \quad |T_1| = 0.0025\\
T_2 = +\,z^5/40 \quad &= -0.0000, \quad |T_2| = 0.000023 < 0.0001 \;\Rightarrow\; \text{stop}.
\end{aligned}$$

Two terms are needed. Sum of accepted terms:

$$S \;=\; -0.2466 + 0.0025 \;=\; -0.2441.$$

With $1/\sqrt{2\pi} = 0.3989$:

$$\Phi(d_1) \;=\; \tfrac{1}{2} + 0.3989 \times (-0.2441) \;=\; 0.5 - 0.0974 \;=\; \boxed{0.4026}.$$

Step 4: Evaluate $\Phi(d_2)$ for $d_2 = -0.3873$

$$\begin{aligned}
z &= -0.3873\\
z^2 &= 0.1500\\
z^3 &= -0.0581\\
z^5 &= -0.0087\\
z^7 &= -0.0013
\end{aligned}$$

$$\begin{aligned}
T_0 &= z &&= -0.3873, \quad |T_0| = 0.3873\\
T_1 &= -\,z^3/6 &&= +0.0097, \quad |T_1| = 0.0097\\
T_2 &= +\,z^5/40 &&= -0.0002, \quad |T_2| = 0.0002\\
T_3 &= -\,z^7/336 &&= +0.0000, \quad |T_3| = 3.9\times 10^{-6} < 0.0001 \;\Rightarrow\; \text{stop.}
\end{aligned}$$

Three terms are needed. Sum of accepted terms:

$$S \;=\; -0.3873 + 0.0097 - 0.0002 \;=\; -0.3778.$$

$$\Phi(d_2) \;=\; \tfrac{1}{2} + 0.3989 \times (-0.3778) \;=\; 0.5 - 0.1507 \;=\; \boxed{0.3493}.$$

Part 1(c) — Compute C by Hand

Continuing the 4-decimal-place rounding convention.

Formula.

$$C \;=\; S_0\,\Phi(d_1) \;-\; K\,e^{-rT}\,\Phi(d_2).$$

Discount factor. With $rT = 0.036 \times 0.523 = 0.0188$, use $e^{-x} = 1 - x + x^2/2 - x^3/6 + \cdots$:

$$\begin{aligned}
x &= 0.0188,\\
x^2 &= (0.0188)^2 = 0.0004,\\
x^2/2 &= 0.0002,\\
x^3 &= 0.0188 \times 0.0004 = 0.0000,\\
e^{-0.0188} &\approx 1 - 0.0188 + 0.0002 - 0.0000 \;=\; 0.9814.
\end{aligned}$$

First term.

$$S_0\,\Phi(d_1) \;=\; 7414 \times 0.4026 \;=\; 2984.8764.$$

Second term.

$$\begin{aligned}
K\,e^{-rT} &\;=\; 7900 \times 0.9814 \;=\; 7753.0600.\\
K\,e^{-rT}\,\Phi(d_2) &\;=\; 7753.0600 \times 0.3493 \;=\; 2708.1439.
\end{aligned}$$

Call price.

$$C \;=\; 2984.8764 - 2708.1439 \;=\; \boxed{\$276.7325}.$$

Part 1(d) — Python Verification

The function `black_scholes_call(S0, K, r, sigma, T)` and a sanity check $C \approx 10.45$ for $S_0 = K = 100, r = 0.05, \sigma = 0.20, T = 1$ are implemented in the companion notebook. The relevant cell:

```
import numpy as np
from scipy.stats import norm

def black_scholes_call(S0, K, r, sigma, T):
    d1 = (np.log(S0/K) + (r + sigma**2/2) * T) / (sigma * np.sqrt(T))
```

```
    d2 = d1 - sigma * np.sqrt(T)
    return S0 * norm.cdf(d1) - K * np.exp(-r*T) * norm.cdf(d2)

# Sanity check (expect ~10.45)
print(black_scholes_call(100, 100, 0.05, 0.20, 1))   # -> 10.4506

# Project parameters
S0, K, r, sigma, T = 7414, 7900, 0.036, 0.1946, 0.523
d1 = (np.log(S0/K) + (r + sigma**2/2)*T) / (sigma*np.sqrt(T))
d2 = d1 - sigma*np.sqrt(T)
print(round(d1, 4), round(d2, 4))                     # -> -0.2470, -0.3877
print(round(norm.cdf(d1), 4), round(norm.cdf(d2), 4))# ->  0.4025,  0.3491
print(black_scholes_call(S0, K, r, sigma, T))        # ->  277.29
```

Comparison 1 — Taylor vs. `scipy`. (All values at 4 decimal places.)

	Taylor series (hand)	`norm.cdf` (Python)	Difference
$\Phi(d_1)$	0.4026	0.4025	0.0001
$\Phi(d_2)$	0.3493	0.3491	0.0002

Both agree to within 0.0002, comfortably better than the three-decimal requirement.

Comparison 2 — C by hand vs. Python.

	Hand C	Python C
Value	\$276.7325	\$277.2896
Difference	\$0.5571 (accumulated 4-dp rounding)	

The \$0.56 gap is the price of rounding every intermediate to four decimal places — chiefly the rounding of $\sigma^2/2 = 0.0189346 \to 0.0190$, which propagates into d_1 and then into both Φ values. Carrying more digits collapses the gap; carrying full precision reproduces \$277.2896 exactly.

Part 1(e) — Compare to the Market Price

Model price C (hand)	Market price C_{market}	Gap (model − market)
\$276.7325	\$180.30	+\$96.4325

The market price is **\$96.43 lower** than the model price — a substantial discount of roughly 35%. Three factors plausibly drive this gap. First, my $\sigma = 19.5\%$ was estimated from eight years of realized daily returns, which include the COVID crash of March 2020 and the 2022 bear market; the option market is forward-looking and is currently pricing in a lower expected volatility over the next six months (*implied volatility* for OTM SPX calls at this maturity is closer to 13–14%). Second, the Black–Scholes formula assumes the index pays no dividends, but SPX has a dividend yield around 1.3%; dividends reduce the forward price of the index and therefore reduce the value of a call — accounting for them would shave several dollars off the model price. Third, there is a well-documented *volatility skew/smile*: out-of-the-money calls (our strike is \$486 above spot) trade at lower implied vols than at-the-money options, because demand concentrates in puts (used as portfolio insurance) and at-the-money calls. Together these effects explain why the market quote of \$180.30 sits well below my Black–Scholes value.