Capstone Project: Modeling with Calculus

Calculus II · STM 1002

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This project uses real market data for educational purposes only and does not constitute investment advice.

This project has four parts. Each has a specific purpose: price a real option using the calculus tools from this course (Part 1), apply the model to a hedging decision (Part 2), interrogate its assumptions (Part 3), and test Black-Scholes against the real world (Part 4). The sequence is deliberate: you should understand a model before you criticize it, and criticize it before you defend it. A short Research Note at the end of the project asks you to communicate your research process for Parts 3 and 4. The process you use here applies any time you're modeling a real-world phenomenon, not just in finance.

The Black-Scholes formula for a European call option:

$$C = S_0\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2),$$

$$d_1 = \frac{\ln(S_0/K) + \left(r + \frac{\sigma^2}{2}\right)T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Symbol	Name	In plain terms
C	Call price	price of the option
S_0	Underlying level	current level of the underlying asset (stock, index, commodity)
K	Strike	asset level at which the option pays off
T	Time to expiry	time to expiration, in years
r	Risk-free rate	interest rate on a riskless investment over the same horizon (our discount rate)
σ	Volatility	how much the underlying asset price moves, annualized; the one parameter you estimate
$\Phi(\cdot)$	Normal CDF	$\Phi(z) = P(Z \le z)$ for standard normal Z
d_1	Moneyness input	determines Δ — how sensitive the option price is to a move in SPX
d_2	Probability input	$\Phi(d_2)$ is the probability the option pays off at expiration

What is a European call option? A call option gives the buyer the right (but not the obligation) to purchase the underlying asset at strike K at expiration *(a put option gives the right to sell at K at expiration)*. "European" means exercise is only possible at expiration, not before. The payoff is $\max(S_T - K, 0)$: positive if S_T closes above K, zero otherwise *(no obligation to exercise)*. Calls are used to speculate on upside, hedge short positions, or cap future purchase costs (as when an airline hedges jet fuel) *(puts serve as portfolio insurance against a price decline)*.

Part 1: Option Pricing, FTC, and Taylor Approximation

In Part 1 you will price a European call option on the S&P 500 index (ticker: `SPX`) using the Black-Scholes model. SPX is a basket of 500 large-cap U.S. companies. The contract has strike $K = 7\,900$ and expires November 20, 2026, paying off only if the index closes above 7 900 on that date.

Checkpoint — Friday, May 15, hard copy: (a) hand computations and answers to all parts below, organized and legible, and (b) a PDF export of your Python notebook.

Project parameters. Ticker: `SPX261120C07900000` (SPX European call, strike 7 900, November 20, 2026 expiry). Use these values throughout the project.

$S_0 = 7\,414$	SPX index level, in index points (live level at data pull time)
$K = 7\,900$	strike, in index points
$T = 0.523$ yr	time to expiration (November 20, 2026 expiry)
$r = 0.036$	T-bill rate (May 13, 2026)
$C_{\text{market}} = \$180.30$	observed option price, in dollars (bid-ask midpoint, May 13, 2026, 11:17 EDT)

A note on units. S_0 and K are in index points; C is in dollars. This is consistent: by convention, one index point equals $1, so the formula gives a dollar price for the option.

(a) **Estimate σ from historical data**

Open the starter notebook at https://tinyurl.com/uatxoptionspart1. The notebook downloads historical SPX closing prices, computes daily log-returns $r_t = \ln(S_t/S_{t-1})$, and annualizes the sample standard deviation by $\sqrt{252}$ to produce σ. Your one decision is the length of the historical window (`lookback_days`). Try a few values, think about the trade-off between stability and recency, and settle on one. Report your estimated σ, the number of returns used, and a brief rationale for your choice.

(b) **Compute $\Phi(d_1)$ and $\Phi(d_2)$ using the Taylor series**

The standard normal CDF is defined as

$$\Phi(z) = \int_{-\infty}^{z} \varphi(t)\,dt, \qquad \text{where } \varphi(t) = \frac{1}{\sqrt{2\pi}}\,e^{-t^2/2}.$$

Your goal is to evaluate $\Phi(d_1)$ and $\Phi(d_2)$ numerically by integrating a Taylor series term by term. Follow the steps below.

(i) Write the Taylor series for e^x centered at $x = 0$. Give the first five terms and the general term $\sum_{n=0}^{\infty}(\cdots)$.

(ii) Substitute $x = -t^2/2$ to obtain a power series for $e^{-t^2/2}$. Write out the first five terms explicitly, simplifying each coefficient.

(iii) Use the series from step (ii) to integrate $\varphi(t)$ and compute $\Phi(z)$. Express your answer in sigma notation.

(iv) First compute d_1 and d_2 using your σ from Part 1(a):

$$d_1 = \frac{\ln(S_0/K) + (r + \sigma^2/2)\,T}{\sigma\sqrt{T}}, \qquad d_2 = d_1 - \sigma\sqrt{T}.$$

Then use your series from step (iii) to evaluate $\Phi(d_1)$ and $\Phi(d_2)$. Use the next-term error bound to determine when to stop: add terms until the next term is smaller than 0.0001 in absolute value. State how many terms you needed for each.

(c) **Compute C by hand**

Using your $\Phi(d_1)$ and $\Phi(d_2)$ from Part 1(b), compute the Black-Scholes call price:

$$C = S_0\,\Phi(d_1) - K\,e^{-rT}\,\Phi(d_2).$$

Show every arithmetic step. Write the formula before substituting numbers.

(d) **Build a Python Black-Scholes function and verify**

In the notebook, complete the `black_scholes_call(S0, K, r, sigma, T)` function using `scipy.stats.norm.cdf` for Φ. A sanity check is already set up: your function should return $C \approx 10.45$ for $S_0 = K = 100$, $r = 0.05$, $\sigma = 0.20$, $T = 1$. Show the sanity check output.

Then run the function on the project parameters. Report two comparisons: (1) your Taylor $\Phi(d_1)$ and $\Phi(d_2)$ from Part 1(b) vs. `norm.cdf(d1)` and `norm.cdf(d2)` — they should agree to at least three decimal places; (2) your hand-computed C from Part 1(c) vs. the Python output — they should agree to within rounding.

(e) **Compare to the market price**

The observed market price for this contract on the data pull date is $C_{\text{market}} = \$180.30$. Compare this to your model price C from Part 1(c). Is the market price higher or lower than your model predicts, and by how much? In 3–4 sentences, suggest what might account for the gap and explain your reasoning, drawing on class discussion, the course readings, or any additional research you find helpful.

Part 2.1: The Greeks and Put-Call Parity

In Part 2.1 you will use calculus tools from Calculus I and Calculus II to study how the Black-Scholes call price responds to changes in its inputs. Traders call these sensitivities *the Greeks*: each Greek is a partial derivative of C with respect to one input, and each measures something specific that someone managing a position needs to know.

Section (a) derives the first Greek, $\Delta = \partial C/\partial S_0$, which captures the sensitivity of the call price to a move in the underlying. Section (b) introduces two more (Vega and Theta) and asks you to compute and interpret them. Section (c) introduces *put-call parity*, a relationship between call and put prices that lets you compute the put from the call. In Part 2.2 (distributed in class on Monday, May 18) you will apply these results to a concrete hedging decision.

Checkpoint — Monday, May 18, hard copy: hand computations and answers to all of Part 2.1, organized and legible. You will receive Part 2.2 along with Parts 3 and 4 in class on Monday.

Use your own σ estimate from Part 1(a) for all numerical computations in Part 2.1.

Part 2.1(a): Deriving Delta

Why does this matter? If you hold one call option, your wealth is exposed to moves in S_0. To hedge that exposure (or even just to know how much risk you are exposed to) you need a number that tells you how much C changes when S_0 changes by one dollar. That number is $\Delta = \partial C/\partial S_0$.

The call price $C = S_0\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$ depends on S_0 in two places: directly through the leading factor S_0, and indirectly through $\Phi(d_1)$ and $\Phi(d_2)$, since d_1 and d_2 are themselves functions of S_0. So at first glance, differentiating C with respect to S_0 looks unpleasant.

However, you'll find that all the messy chain-rule pieces *exactly cancel*, leaving something unexpectedly clean. The cancellation is not obvious from the formula. It depends on a particular algebraic identity between d_1 and d_2. Your job in steps (i)–(v) below is to make the cancellation happen and see what falls out.

Recall (Block 1).

Name	**Function**	**What it gives**
$\varphi(z)$, standard normal PDF	$\dfrac{1}{\sqrt{2\pi}}\,e^{-z^2/2}$	probability density at z
$\Phi(z)$, standard normal CDF	$\displaystyle\int_{-\infty}^{z} \varphi(t)\,dt$	accumulated probability up to z

(i) Apply the product rule to $C = S_0\,\Phi(d_1) - Ke^{-rT}\,\Phi(d_2)$, differentiating with respect to S_0. Write all resulting terms before any simplification.

(ii) Compute $\partial d_1/\partial S_0$.

Hint. From the definition, $d_1 = \left[\ln(S_0/K) + (r + \sigma^2/2)\,T\right]/(\sigma\sqrt{T})$. Only the $\ln(S_0/K)$ piece depends on S_0.

Then show that $\dfrac{\partial d_2}{\partial S_0} = \dfrac{\partial d_1}{\partial S_0}$. (Since $d_2 = d_1 - \sigma\sqrt{T}$ and the second term does not depend on S_0, this should be quick.)

(iii) Substitute your result from step (ii) into step (i). Factor to obtain

$$\frac{\partial C}{\partial S_0} = \Phi(d_1) + \frac{S_0\,\varphi(d_1) - Ke^{-rT}\,\varphi(d_2)}{S_0\,\sigma\sqrt{T}}.$$

(iv) Prove the identity $S_0\,\varphi(d_1) = Ke^{-rT}\,\varphi(d_2)$, which forces the numerator in step (iii) to vanish.

Hint. Take logs of both sides; then $d_1^2 - d_2^2$ factors.

(v) Conclude from steps (iii)–(iv) that $\Delta = \Phi(d_1)$. Compute the numerical value of Δ for the project option using your $\Phi(d_1)$ from Part 1(b). State what Δ means: for a one-point rise in S_0, by how much does the call price change?

The Greeks: reference for Parts 2.1(b) and 2.2. Each Greek is a partial derivative of C with respect to one input. The formulas can all be derived by the same kind of partial differentiation you used in Part 2.1(a).

Greek	Formula	What it measures (long call)
$\Delta = \dfrac{\partial C}{\partial S_0}$	$\Phi(d_1)$	change in C for a 1-point rise in S_0
$\Gamma = \dfrac{\partial^2 C}{\partial S_0^2}$	$\dfrac{\varphi(d_1)}{S_0\,\sigma\sqrt{T}}$	rate of change of Δ
$\text{Vega} = \dfrac{\partial C}{\partial \sigma}$	$S_0\,\sqrt{T}\,\varphi(d_1)$	change in C per unit rise in σ (formula gives the raw derivative with σ as a decimal; see Part 2.1(b) for the trader's convention)
$\Theta = \dfrac{\partial C}{\partial t}$	$-\dfrac{S_0\,\sigma\,\varphi(d_1)}{2\sqrt{T}} - r\,K\,e^{-rT}\,\Phi(d_2)$	change in C per year of calendar time t (formula in dollars/year; see Part 2.1(b) for the trader's convention)

Part 2.1(b): Vega and Theta

Using the Vega and Theta formulas from the reference block above, compute the numerical values for the project option (use your σ, d_1, d_2 from Part 1). For each:

(i) **Raw and conventional values.** The formulas as written give raw derivatives, but traders quote each Greek in a specific convention. Convert each as follows:

- **Vega.** $S_0\sqrt{T}\,\varphi(d_1)$ gives change in C per *unit* rise in σ. Multiply by 0.01 to get Vega per one-percentage-point (0.01) rise in σ.
- **Theta.** The formula gives change in C per *year* of calendar time. Divide by 365 to get Theta per calendar day.

Report both the raw value and the converted value, with units labeled.

(ii) Write one sentence interpreting the sign of each.

(iii) Compare the conventional magnitudes to the call price C from Part 1(c). Is this option particularly sensitive to volatility? To the passage of time? Write one sentence for each.

Part 2.1(c): Put-Call Parity

Put-call parity relates the prices of a European call and a European put with the same strike and expiration:

$$P = C - S_0 + Ke^{-rT}.$$

This formula follows from a no-arbitrage argument we worked through in class: a portfolio that is long one call, short one unit of the underlying, and long a zero-coupon bond paying K at expiration replicates a long put exactly, so the market enforces $P + S_0 = C + Ke^{-rT}$. The argument uses no assumption about how stock prices evolve. The absolute prices C and P depend on the model; the relationship between them does not.

Note on dividends. This form of parity assumes the underlying pays no dividends. SPX carries an effective dividend yield of roughly 1.4% per year from its constituent stocks; the strictly correct version replaces S_0 with S_0e^{-qT} where q is the dividend yield. To simplify modeling in this project, we assume no dividends.

(i) Compute P using C from Part 1(c). Show the arithmetic. Verify that $P > 0$.

(ii) The call we just priced gains value on a rally. A long-equity investor who fears a decline needs an instrument that gains value when the index *falls* — not this one.

   (a) Identify the correct hedging instrument for a long-equity investor and explain in 2–3 sentences why.

   (b) Despite calls being the wrong hedge for portfolio insurance, OTM SPX calls trade with reasonable liquidity. Name a couple of possible buyer types whose use of the call is *not* portfolio insurance, and briefly describe what each is doing.

Part 2.2: Portfolio Protection (Application)

Distributed in class Monday, May 18.

Apply the results from Part 2.1 to a concrete hedging decision.

You hold a long position of 200 units of SPX index exposure. You are concerned about a significant index decline and want to limit your downside without exiting the position. To hedge, you buy 200 of the $K = 7\,900$ puts whose price you computed in Part 2.1(c), one per unit of index exposure.

Recall that Delta measures how much a position's value changes for a one-point move in the underlying, which makes it the standard tool for sizing risk exposure and for building hedges that offset it.

(a) **Hedged portfolio Delta**

(i) **Original position Delta.** Each unit of underlying exposure moves dollar-for-dollar with the index, so its value is simply S_0 and its Delta is $\partial S_0/\partial S_0 = 1$. State the Delta of your 200-unit position.

(ii) **Position Delta of the puts.** The Delta of a single European put is $\Delta_{\text{put}} = \Phi(d_1) - 1$ (this follows from differentiating the parity relation $P = C - S_0 + Ke^{-rT}$ with respect to S_0). Use the $\Phi(d_1)$ from your Part 2.1(a). State Δ_{put} numerically and interpret the sign and magnitude.

(iii) **Apply linearity.** The partial derivative $\partial/\partial S_0$ distributes over sums and constant multiples, just like an ordinary derivative. So the Delta of any portfolio is the sum of position Deltas, each weighted by the number of units held:

$$\Delta_{\text{portfolio}} = \sum_i n_i \, \Delta_i.$$

Apply this rule to the hedged portfolio (200 units of underlying + 200 puts) to compute $\Delta_{\text{portfolio}}$ numerically. Then interpret: for a one-point rise in SPX, by how much does the hedged portfolio value change? Compare to the unhedged exposure of 200.

(b) **Delta-neutral**

A portfolio is **delta-neutral** when its net Delta is zero: small moves in the underlying do not change its value (to first order).

Traders aim for delta-neutral when they want their P&L to depend on something other than the direction of the underlying. The classic example is market making: a dealer who quotes bid and ask prices on options profits from the bid-ask spread, not from betting on which way the market goes. After each option trade, the dealer's inventory has some Delta; they buy or sell the underlying to neutralize it. Another example is volatility trading: a trader who thinks implied volatility is too high (or too low) can build a delta-neutral position whose value depends on volatility rather than direction. The general idea: zero out the Greek you don't want to bet on, so your P&L tracks the Greek you do.

Suppose you keep your 200 units of underlying exposure but are free to choose the number of puts. How many of these puts (N) would you need to hold to make the portfolio delta-neutral? Set up the equation, solve for N, and state your answer numerically. State your $\Phi(d_1)$ from Part 2.1(a) explicitly alongside your answer so we can compare across students on Friday. Compare to the 200-put hedge in (a): roughly how much more (or less) optionality does the delta-neutral hedge require?

Note that holding more puts also brings more portfolio Vega and Theta exposure, so full delta-neutralization is not free.

Part 3: Model Critique

I have not assigned a reading for this part. I expect you to research these topics yourself. You may use any sources you find useful, including AI tools. Document your sources and process in the Research Note at the end of the project (see below). The analysis and writing must represent your own engagement and conclusions. Remember, your report is the subject of a graded class discussion on Friday, May 22.

No numerical computation is required in this part. Write clearly and concisely; depth matters more than length. Where relevant, draw on your work in Parts 1 and 2.

(a) **Two violated assumptions**

Name two assumptions of Black-Scholes that are routinely violated in real markets. For each, describe in 2–3 sentences how the violation affects option prices or risk estimates. (Part 4 is where you will tie these to specific real-world episodes; here, focus on the mechanism.)

(b) **Predecessors**

Pick one method that was used to price options before Black-Scholes (1973). Candidates include Louis Bachelier's 1900 thesis (*Théorie de la spéculation*, arithmetic Brownian motion), Sprenkle (1961), Boness (1964), or trader rule-of-thumb pricing. In 3–5 sentences: describe what the method did, and name one specific thing about it that Black-Scholes improved on.

(c) **Successors**

Pick one model or framework that came after Black-Scholes and was designed to address one of its limitations. Candidates include the Heston stochastic volatility model, the Merton jump-diffusion model, Dupire's local volatility model, or the modern implied volatility surface as an empirical correction. In 3–5 sentences: what BS limitation does it address, how does it do so (describe the mechanism in words, no equations), and what does it cost in complexity or calibration?

Part 4: Black-Scholes in the Real World

In this section, you'll think about Black-Scholes in the context of the real world. Where has it failed, and where is it the right tool? By the end, you'll need to state and defend your own view on when Black-Scholes-style modeling is appropriate, supported by specific evidence.

Write approximately two pages total across the three subsections below. Name episodes and settings specifically (dates, institutions, what was at stake, what happened). Where relevant, connect back to your Part 3 answers.

(a) Critique

Research the 1998 collapse of Long-Term Capital Management. Describe what happened and what was at stake. Then identify one or two specific Black-Scholes assumptions that contributed to the failure and explain the mechanism, not just that the model was wrong but why that particular assumption mattered in that particular context.

(b) Defense

Black-Scholes implied volatility functions as the standard quoting language between counterparties in liquid vanilla options markets. Explain what this means, and why in this specific setting the model's simplicity and transparency are advantages, not just acceptable tradeoffs but reasons a more sophisticated model would not necessarily be better.

(c) Take a Position

In no more than two paragraphs, state your own view: under what conditions is Black-Scholes-style modeling appropriate, and under what conditions is it not? Refer back to both your critique and your defense. Make a real claim that someone could disagree with, and give the reasoning behind it.

Document your sources in the Research Note at the end of the project.

Research Note

A short note placed at the end of your project document. Graded as part of Parts 3 and 4: a deduction applies if it is missing or thin. The job of this note is to communicate your research process. Three short labeled items, 1–3 sentences each:

- **Sources used.** List the articles, books, and AI tools you consulted for Parts 3 and 4. For each AI tool, write one sentence on how you used it: for example, "to summarize the LTCM collapse," "to draft an initial paragraph that I then revised," or "to suggest model extensions I could choose from in 3(c)."

- **One claim cross-checked or rejected.** Name one specific claim that came up in your research, and either (i) how you verified it across two sources, or (ii) why you chose not to include it because you could not verify it.

- **Brief reflection.** Two to three sentences on how the sources shaped your conclusions, what was hardest to verify, or what you learned about using these tools for this kind of research.

In-Class Discussion (Fri 5/22)

The final report is due **Thursday, May 21 by midnight on Populi**. On Friday, May 22 we will hold a graded in-class discussion in two parts.

Individual opening (∼3 min each): Each student will be asked to explain one specific item from their submitted work. You will not know in advance which item will be chosen, so the best preparation is to understand all of your work.

Group discussion (~35 min): We will then move to an open seminar anchored on a question drawing on everyone's results. Since you all worked with the same option parameters, you can compare σ estimates, model prices, and analytical conclusions directly. Come ready to discuss, argue, and respond to each other.

You may refer to your submitted document during both portions.