Capstone Project Part 2.1: The Greeks and Put–Call Parity

Nathaniel Freed

May 18, 2026

Project Parameters (same as Part 1)

$S_0 = 7414$	SPX index level (May 13, 2026)
$K = 7900$	strike (index points)
$T = 0.523$	years to expiry (Nov 20, 2026)
$r = 0.036$	13-week T-bill rate
$C_{\text{market}} = \$180.30$	bid-ask midpoint

Inputs reused from Part 1 (all hand-rounded to 4 dp). $\sigma = 0.1946$, $\sqrt{T} = 0.7232$, $\sigma\sqrt{T} = 0.1407$, $rT = 0.0188$, $e^{-rT} = 0.9814$, $d_1 = -0.2466$, $d_2 = -0.3873$, $\Phi(d_1) = 0.4026$, $\Phi(d_2) = 0.3493$, $C = \$276.7325$.

Every intermediate calculation below is rounded to four decimal places before being used in the next step (same convention as Part 1).

Part 2.1(a) — Deriving Delta

(i) Product rule applied to C with respect to S_0

$C = S_0\,\Phi(d_1) \;-\; K\,e^{-rT}\,\Phi(d_2)$ depends on S_0 directly through the leading factor and indirectly through d_1, d_2. Differentiating term-by-term, with Ke^{-rT} treated as a constant:

$$\frac{\partial C}{\partial S_0} \;=\; \underbrace{\frac{\partial S_0}{\partial S_0}}_{=1}\Phi(d_1) \;+\; S_0 \cdot \frac{\partial \Phi(d_1)}{\partial S_0} \;-\; Ke^{-rT} \cdot \frac{\partial \Phi(d_2)}{\partial S_0}.$$

Applying the chain rule with $\Phi'(z) = \varphi(z)$:

$$\boxed{\frac{\partial C}{\partial S_0} \;=\; \Phi(d_1) \;+\; S_0\,\varphi(d_1)\,\frac{\partial d_1}{\partial S_0} \;-\; Ke^{-rT}\,\varphi(d_2)\,\frac{\partial d_2}{\partial S_0}.}$$

All resulting terms are written before any simplification.

(ii) Compute $\partial d_1/\partial S_0$ and show $\partial d_1/\partial S_0 = \partial d_2/\partial S_0$

From the definition,

$$d_1 \;=\; \frac{\ln(S_0/K) \;+\; (r + \sigma^2/2)\,T}{\sigma\sqrt{T}}.$$

The denominator $\sigma\sqrt{T}$ and the term $(r+\sigma^2/2)T$ do not depend on S_0. Only $\ln(S_0/K) = \ln S_0 - \ln K$ depends on S_0, and $\dfrac{d}{dS_0}\ln S_0 = \dfrac{1}{S_0}$. Therefore

$$\boxed{\frac{\partial d_1}{\partial S_0} \;=\; \frac{1}{S_0\,\sigma\sqrt{T}}.}$$

For d_2: $d_2 = d_1 - \sigma\sqrt{T}$, and $\sigma\sqrt{T}$ does not depend on S_0, so

$$\frac{\partial d_2}{\partial S_0} \;=\; \frac{\partial d_1}{\partial S_0} \;-\; 0 \;=\; \frac{1}{S_0\,\sigma\sqrt{T}}.$$

Hence $\dfrac{\partial d_1}{\partial S_0} = \dfrac{\partial d_2}{\partial S_0}$.

(iii) Substitute into (i) and factor

Inserting the common value $1/(S_0\,\sigma\sqrt{T})$ into the chain-rule expression from (i):

$$\frac{\partial C}{\partial S_0} \;=\; \Phi(d_1) \;+\; S_0\,\varphi(d_1)\cdot\frac{1}{S_0\,\sigma\sqrt{T}} \;-\; Ke^{-rT}\,\varphi(d_2)\cdot\frac{1}{S_0\,\sigma\sqrt{T}}.$$

Factor $1/(S_0\,\sigma\sqrt{T})$ from the last two terms:

$$\boxed{\frac{\partial C}{\partial S_0} \;=\; \Phi(d_1) \;+\; \frac{S_0\,\varphi(d_1) \;-\; Ke^{-rT}\,\varphi(d_2)}{S_0\,\sigma\sqrt{T}}.}$$

(iv) Prove the identity $S_0\,\varphi(d_1) = Ke^{-rT}\,\varphi(d_2)$

Take logs of both sides. Using $\varphi(z) = \dfrac{1}{\sqrt{2\pi}}e^{-z^2/2}$ so that $\ln\varphi(z) = -\tfrac{1}{2}\ln(2\pi) - z^2/2$:

$$\ln S_0 \;-\; \tfrac{1}{2}\ln(2\pi) \;-\; \frac{d_1^2}{2} \;\stackrel{?}{=}\; \ln K \;-\; rT \;-\; \tfrac{1}{2}\ln(2\pi) \;-\; \frac{d_2^2}{2}.$$

The $-\tfrac{1}{2}\ln(2\pi)$ terms cancel. Rearranging,

$$\ln(S_0/K) \;+\; rT \;\stackrel{?}{=}\; \frac{d_1^2 - d_2^2}{2}. \qquad (\star)$$

Now factor $d_1^2 - d_2^2 = (d_1 - d_2)(d_1 + d_2)$. Since $d_1 - d_2 = \sigma\sqrt{T}$ and $d_1 + d_2 = 2d_1 - \sigma\sqrt{T}$,

$$d_1^2 - d_2^2 \;=\; \sigma\sqrt{T}\left(2d_1 - \sigma\sqrt{T}\right) \;=\; 2\,\sigma\sqrt{T}\,d_1 \;-\; \sigma^2 T.$$

Substitute the definition $\sigma\sqrt{T}\,d_1 = \ln(S_0/K) + (r+\sigma^2/2)\,T$:

$$d_1^2 - d_2^2 \;=\; 2\big[\ln(S_0/K) + (r+\sigma^2/2)\,T\big] \;-\; \sigma^2 T \;=\; 2\ln(S_0/K) \;+\; 2rT \;+\; \sigma^2 T \;-\; \sigma^2 T \;=\; 2\ln(S_0/K) \;+\; 2rT.$$

Dividing by 2:

$$\frac{d_1^2 - d_2^2}{2} \;=\; \ln(S_0/K) \;+\; rT.$$

This is exactly $(\star)$, so the identity holds. Therefore $S_0\,\varphi(d_1) = Ke^{-rT}\,\varphi(d_2)$, and the numerator in (iii) vanishes.

(v) Conclude $\Delta = \Phi(d_1)$ and compute numerically

From (iii) and (iv), the second term is zero, leaving

$$\boxed{\Delta \;=\; \frac{\partial C}{\partial S_0} \;=\; \Phi(d_1).}$$

Using $\Phi(d_1) = 0.4026$ from Part 1(b):

$$\boxed{\Delta \;=\; 0.4026.}$$

Interpretation. For a 1-index-point rise in S_0, the Black–Scholes call price rises by approximately \$0.4026. Equivalently, holding one call exposes the holder to about 40.3% of the dollar move in SPX — the call is roughly 40% as sensitive to the index as one share of the underlying.

Part 2.1(b) — Vega and Theta

Setup: compute $\varphi(d_1)$ via Taylor series

$\varphi(d_1) = \dfrac{1}{\sqrt{2\pi}}\, e^{-d_1^2/2}$. With $d_1 = -0.2466$:

$$\begin{aligned} d_1^2 &= (-0.2466)^2 = 0.0608, \\ d_1^2/2 &= 0.0608/2 = 0.0304. \end{aligned}$$

Evaluate $e^{-0.0304}$ via $e^{-x} = 1 - x + x^2/2 - x^3/6 + \cdots$ with $x = 0.0304$:

$$\begin{aligned} x &= 0.0304, \\ x^2 &= (0.0304)^2 = 0.0009, \\ x^2/2 &= 0.0005, \\ x^3 &= 0.0304 \times 0.0009 = 0.0000, \\ e^{-0.0304} &\approx 1 - 0.0304 + 0.0005 - 0.0000 \;=\; 0.9701. \end{aligned}$$

With $1/\sqrt{2\pi} = 0.3989$:

$$\varphi(d_1) \;=\; 0.3989 \times 0.9701 \;=\; \boxed{0.3870}.$$

(i) Vega: raw and conventional

Formula. $\text{Vega} = \dfrac{\partial C}{\partial \sigma} = S_0\,\sqrt{T}\,\varphi(d_1)$.

$$\begin{aligned} S_0\sqrt{T} &= 7414 \times 0.7232 \;=\; 5361.8048, \\ \text{Vega (raw)} \;=\; S_0\sqrt{T}\,\varphi(d_1) &= 5361.8048 \times 0.3870 \;=\; \boxed{2075.0185} \quad \text{(call-price points per 1.0 rise in } \sigma). \end{aligned}$$

Conventional Vega (per one percentage point, i.e. 0.01 rise in σ).

$$\text{Vega (per 1pp)} \;=\; 2075.0185 \times 0.01 \;=\; \boxed{20.7502} \quad \text{(call-price points per 0.01 rise in } \sigma).$$

(i) Theta: raw and conventional

Formula. $\Theta = \dfrac{\partial C}{\partial t} \;=\; -\dfrac{S_0\,\sigma\,\varphi(d_1)}{2\sqrt{T}} \;-\; r\,K\,e^{-rT}\,\Phi(d_2).$

First term: $-\,S_0\,\sigma\,\varphi(d_1)/(2\sqrt{T})$.

$$\begin{aligned}
S_0\,\sigma &= 7414 \times 0.1946 \;=\; 1442.7644,\\
S_0\,\sigma\,\varphi(d_1) &= 1442.7644 \times 0.3870 \;=\; 558.3498,\\
2\sqrt{T} &= 2 \times 0.7232 \;=\; 1.4464,\\
\text{first term} &= -\,558.3498\,/\,1.4464 \;=\; -386.0272.
\end{aligned}$$

Second term: $-\,r\,K\,e^{-rT}\,\Phi(d_2)$.

$$\begin{aligned}
r\,K &= 0.036 \times 7900 \;=\; 284.4000,\\
r\,K\,e^{-rT} &= 284.4000 \times 0.9814 \;=\; 279.1102,\\
r\,K\,e^{-rT}\,\Phi(d_2) &= 279.1102 \times 0.3493 \;=\; 97.4932,\\
\text{second term} &= -\,97.4932.
\end{aligned}$$

Raw Theta.

$$\Theta \;=\; -386.0272 \;+\; (-97.4932) \;=\; \boxed{-483.5204} \quad \text{(call-price points per year of calendar time).}$$

Conventional Theta (per calendar day). Divide by 365:

$$\Theta_{\text{day}} \;=\; -483.5204\,/\,365 \;=\; \boxed{-1.3247} \quad \text{(call-price points per calendar day).}$$

(ii) Sign interpretation

Vega. Vega > 0: a rise in volatility increases the call's value, because greater uncertainty in S_T makes the optionality payoff $\max(S_T - K, 0)$ more valuable (more probability mass at high S_T, and the floor at zero limits downside).

Theta. $\Theta < 0$: the call loses value as calendar time passes, because shrinking time-to-expiry leaves fewer opportunities for S to drift above $K = 7900$ before expiration — this is the time-decay every long-option holder pays.

(iii) Magnitude comparison to $C = \$276.7325$

Volatility sensitivity. $\dfrac{|\text{Vega (per 1pp)}|}{C} = \dfrac{20.7502}{276.7325} \approx 0.0750$, i.e. a 1-percentage-point shift in σ moves the call's value by roughly 7.5% — this option is materially sensitive to volatility.

Time sensitivity. $\dfrac{|\Theta_{\text{day}}|}{C} = \dfrac{1.3247}{276.7325} \approx 0.0048$, i.e. the call loses about 0.48% of its value per calendar day at current parameters — meaningful but moderate; over the ~191 calendar days to expiry the cumulative time decay is a sizable fraction of the premium, so this option is also sensitive to the passage of time.

Part 2.1(c) — Put–Call Parity

(i) Compute P and verify $P > 0$

Formula. $P \;=\; C \;-\; S_0 \;+\; K\,e^{-rT}$.

Inputs from Part 1(c): $C = 276.7325$, $K\,e^{-rT} = 7900 \times 0.9814 = 7753.0600$.

$$C - S_0 = 276.7325 - 7414.0000 \;=\; -7137.2675,$$
$$P \;=\; (C - S_0) + Ke^{-rT} = -7137.2675 + 7753.0600 \;=\; \boxed{615.7925}.$$

Since $615.7925 > 0$, the parity-implied put price is positive, as required. ✓

(ii)(a) Correct hedging instrument for a long-equity investor

The correct hedge is a **long put** on the index. A put's payoff is $\max(K - S_T, 0)$, so it pays off precisely when the index falls below the strike — which is when the long-equity position is losing money. Holding a put alongside a long-equity position therefore caps the downside while leaving the upside intact. Long index puts (typically slightly out-of-the-money SPX puts) are the standard portfolio-insurance instrument; calls move the wrong way for someone who fears a decline.

(ii)(b) Non-insurance buyers of OTM SPX calls

OTM SPX calls trade liquidly despite being the wrong portfolio-insurance vehicle. Two buyer types whose use of the call is *not* portfolio insurance:

- **Directional speculators (leveraged bullish bets).** A trader who expects SPX to rally above 7900 by November can buy this call for \$180.30 of premium and obtain convex exposure to roughly 7900 index points of notional. The capped downside (the premium) and unbounded upside make calls a cheap, leveraged way to express a bullish directional view — no portfolio to insure, just a positioning trade.
- **Volatility traders / dealer desks running long-gamma or long-vega books.** A delta-hedged long call has near-zero directional exposure but pays off when realized volatility exceeds implied (via gamma scalping) and benefits from a rise in implied vol (positive Vega). For these traders the call is a vehicle for trading Γ and Vega, not direction; the underlying delta is hedged away with a short position in SPX futures.

(Other plausible buyers include covered-call writers buying back previously-sold calls to close short positions, and option market-makers warehousing inventory; the two above are the most common non-insurance use cases.)